Langevin equations and surface growth

G. Costanza
Departamento de Física, Universidad Nacional de San Luis, Chacabuco 917, 5700 San Luis, Argentina
(Received 9 January 1997)

The three approaches usually used to study surface growth are (i) master equation, (ii) stochastic Langevin equation, and (iii) microscopic models. All of them give the same scaling exponents. Recently, Vvedensky *et al.* [Phys. Rev. E **48**, 852 (1993)] derived a stochastic Langevin equation from a master equation of the birth and death type, for the epitaxial growth, demonstrating the equivalence of both approaches. In this paper a stochastic Langevin equation is derived from a discrete model. The results are the same as those obtained by Vvedensky *et al.* demonstrating that the three approaches are equivalent. As a nontrivial example, our procedure is used to derive the Kardar-Parisi-Zhang (KPZ) equation from the ballistic deposition process. This model with vacancies and overhangs is very difficult to handle, due to the algebraic complications that arise when the master equation approach is used. [S1063-651X(97)11905-5]

PACS number(s): 05.40.+j

I. INTRODUCTION

In recent years there has been a growing interest in the study of surface growth. The comprehension of this phenomenon plays an important role in understanding and controlling many interesting interface processes [1], such as colloidal and porous aggregates [2–4], vapor deposition [5–7], crystal growth, molecular beam epitaxy (MBE) [8], etc. In the laboratory, these processes can be made to occur in many different ways and a complete understanding is still not available.

The formation of interfaces and surfaces is influenced by a large number of factors, and it is almost impossible to distinguish all of them. Several studies show that some properties of growing surfaces can be analyzed and described by simple microscopical rules. Many models can afford to be elementary, because the phenomena being modeled are themselves rather simple, i.e., ''island formation'' in MBE can be modeled successfully using elementary computer algorithms.

Nevertheless, the classical procedure to describe such processes from the mathematical point of view consists of the following: (i) a master equation to determine the joint probability $P(h,t)$ that the surface profile has a configuration h at time t [where $h=(h_1,h_2,\dots,h_N)$ is the set of different surface sites height]; (ii) a stochastic equation of the Langevin type to determine $h(x_i,t)$, which describes the height h in the position x_i at time t; (iii) the discrete model, which describes through specific rules the growth of the height $h_i(t_n)$ of a given site i in the time step t_n. Given a model for the description of a growing process the three mentioned methods give the same scaling exponent; the roughening exponent α, the growth exponent β, and the dynamic exponent $z=\alpha/\beta$.

Vvedensky *et al.* [9] have reported an analytical derivation of the Langevin equation (LE) of motion for the surface of a solid growing under typical epitaxial conditions. By using the master equation for the growing process, they obtained the Langevin equation through the evaluation of the first moment of the transition rate $W(h,h')$. Both equations belong to the same universality class and as a consequence they have the same scaling exponent. While in principle the method described by the authors can be applied to the more general case (i.e., surface growth with vacancies and overhangs), the algebraic complications that arise are daunting, as they mention in their paper [9]. Due to this fact, it is interesting and stimulating to search for alternative approaches to describe the growing process.

In this paper, we will use a microscopic approach to obtain the Langevin equation for the motion of the surface profile in a growing process. The outline of the rest of this paper is as follows; in Sec. II A, we describe the procedure to obtain the LE from elementary rules for the discrete Das Sarma–Tamborenea growth model [10–12] (without diffusion) obtaining the corresponding Edwards-Wilkinson [13] equation and in Sec. II B the ballistic deposition model is analyzed and the corresponding Kardar-Parisi-Zhang [14] equation is obtained. Finally, in Sec. III we give a brief summary and conclusion of the present investigation.

II. DISCRETE MODELS AND LANGEVIN-TYPE EQUATIONS

Our procedure to obtain the Langevin equation for the motion of the surface profile is based on the elementary microscopic growth rules for the height of a given site. In the following we consider a one-dimensional lattice with N sites with periodic boundary conditions. The height of a given site i, $h_i(t_n)$, is a function of the index i and time t_n. One can specify the procedure as follows:

(i) A dummy index j is chosen at random from N integer numbers. (ii) The height of a site i at time $t_{n+1}=t_n+\tau_0$ (here τ_0 is the elemental time step between two successive depositions in any site of the lattice) is given by

$$h_i(t_{n+1})=R_j(\{h_i(t_n)\}), \tag{1}$$

where $R_j(\{h_i(t_n)\})$ gives the growing rules for the height of the site i and depends on the value of the dummy index j. The way to obtain such dependence is specified by the rules *a priori*, and in general can be dependent on the complete set of heights $\{h_i(t_n)\}$ at time t_n before deposition.

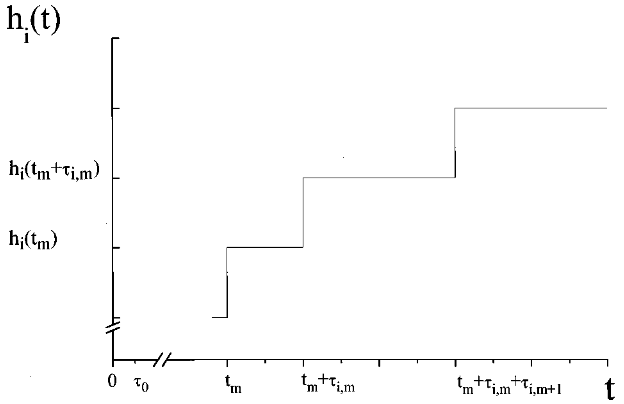

FIG. 1. A portion of the $h_i(t)$ plot. It is easy to see that, due to the fact that $h_i(t)$ is constant between two successive depositions, $h_i(t_m+\tau_{i,m}-\tau_0)=h_i(t_m)$.

Let us define now the time interval between two successive depositions on site i as

$$\tau_{i,m}=\tau+\delta\tau_{i,m}, \tag{2}$$

where τ is the mean time interval between two successive depositions on site i and $\delta\tau_{i,m}$ is the random deviation from the mean value τ. With this consideration we can write Eq. (1) for time $t+\tau_{i,m}$ as

$$h_i(t+\tau_{i,m})=R_j(\{h_i(t+\tau_{i,m}-\tau_0)\}), \tag{3}$$

where $t=t_m$ is the time of the previous deposition (see Fig. 1).

In order to illustrate the derivation of the LE from the growth rules given in Eq. (3), in the next section we will obtain the rules for two well-known growth models: (1) the Das Sarma–Tamborenea models (without diffusion) and (2) the ballistic deposition [1] and consequently the corresponding Langevin equations.

A. Das Sarma–Tamborenea model (without diffusion)

The first nonequilibrium growth models including deposition, relaxation, and surface diffusion were introduced by Family [10], Wolf and Villain [11], Das Sarma and Tamborenea [12], etc., in order to mimic real growth processes. In this subsection we apply our procedure to obtain the Edwards-Wilkinson [13] equation from the elementary relaxation mechanisms of the Das Sarma–Tamborenea (DST) model (without diffusion). Before proceeding to define the elementary rules, let us define the height difference H_l^k between two sites k and l at time $t+\tau_{i,n}-\tau_0$ as

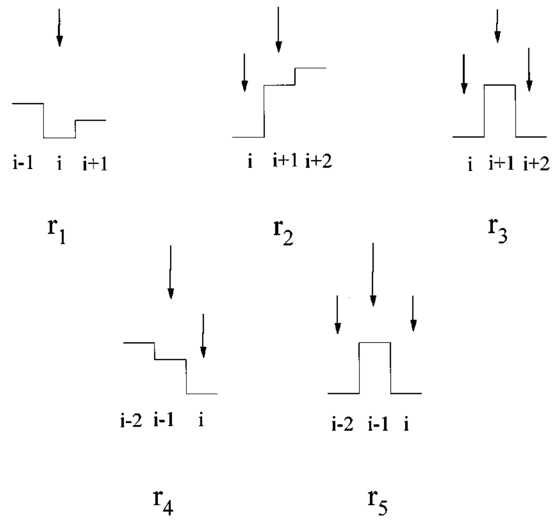

FIG. 2. The five graphical representations corresponding to the rules $r_1,\ldots,r_5$, indicating all the possible situations of site i growth. The upper arrows indicate the randomly chosen site j and the lower arrows indicate the possible sites where the falling particles relax.

$$H_l^k=h_k(t+\tau_{i,n}-\tau_0)-h_l(t+\tau_{i,n}-\tau_0). \tag{4}$$

By expanding H_l^k in a Taylor series about t and retaining the first term, we obtain

$$H_l^k=h_k(t)-h_l(t)+O(\tau_{i,n}-\tau_0), \tag{5}$$

which is an approximation that will be used throughout this paper.

Now, we have to derive explicitly the rules given in Eq. (3) for the DST model. The growing rules for the site i (see Fig. 2) can be expressed as conditionals in the following way:

$$h_i(t+\tau_{i,n})=h_i(t)+r_1+r_2+r_3+r_4+r_5, \tag{6}$$

where the elementary rules $r_1,\ldots,r_5$ can be written as

$$r_1=\begin{cases} a & \text{if} \quad j=i \quad \text{and} \quad H_i^{i-1}\geqslant 0 \quad \text{and} \quad H_i^{i+1}\geqslant 0\\ 0 & \text{if} \quad j\neq i \quad \text{or} \quad H_i^{i-1}<0 \quad \text{or} \quad H_i^{i+1}>0, \end{cases} \tag{7}$$

$$r_2=\begin{cases} a & \text{if} \quad j=i+1 \quad \text{and} \quad H_i^{i+1}>0 \quad \text{and} \quad H_{i+1}^{i+2}\geqslant 0\\ 0 & \text{if} \quad j\neq i+1 \quad \text{or} \quad H_i^{i+1}\leqslant 0 \quad \text{or} \quad H_{i+1}^{i+2}<0, \end{cases} \tag{8}$$

$$r_3=\begin{cases} a/2 & \text{if} \quad j=i+1 \quad \text{and} \quad H_i^{i+1}>0 \quad \text{and} \quad H_{i+1}^{i+2}<0\\ 0 & \text{if} \quad j\neq i+1 \quad \text{or} \quad H_i^{i+1}\leqslant 0 \quad \text{or} \quad H_{i+1}^{i+2}\leqslant 0, \end{cases} \tag{9}$$

$$r_4=\begin{cases} a & \text{if} \quad j=i-1 \quad \text{and} \quad H_{i-1}^{i-2}\geqslant 0 \quad \text{and} \quad H_i^{i-1}>0\\ 0 & \text{if} \quad j\neq i-1 \quad \text{or} \quad H_{i-1}^{i-2}<0 \quad \text{or} \quad H_i^{i-1}\leqslant 0, \end{cases} \tag{10}$$

$$r_5=\begin{cases} a/2 & \text{if} \quad j=i-1 \quad \text{and} \quad H_{i-2}^{i-1}>0 \quad \text{and} \quad H_i^{i-1}>0 \\ 0 & \text{if} \quad j\neq i-1 \quad \text{or} \quad H_{i-2}^{i-1}\leqslant 0 \quad \text{or} \quad H_i^{i-1}\leqslant 0, \end{cases} \tag{11}$$

and $h_i(t+\tau_{i,n}-\tau_0)=h_i(t)$ was used (see Fig. 1). The equivalent analytic expression of Eq. (6) is given by

$$h_i(t+\tau_{i,n})=h_i(t)+a[\delta_{i,j}\theta(H_i^{i-1})\theta(H_i^{i+1})+\delta_{i+1,j}\theta^*(H_i^{i+1})\theta(H_{i+1}^{i+2})+\tfrac{1}{2}\delta_{i+1,j}\theta^*(H_i^{i+1})\theta^*(H_{i+2}^{i+1}) + \delta_{i-1,j}\theta(H_{i-1}^{i-2})\theta^*(H_i^{i-1})+\tfrac{1}{2}\delta_{i-1,j}\theta^*(H_{i-2}^{i-1})\theta^*(H_i^{i-1})] \tag{12}$$

where $\theta(x)$ is the step function and $\theta^*(x)=1-\theta(-x)$. The products in each terms of Eq. (12) are the analytic expression of the corresponding conditionals given in Eqs. (7–11). The factor $\frac{1}{2}$ in the third and fifth terms of Eq. (12) is the probability that a particle deposits on site i when the dummy random index j takes the value $i+1$ or $i-1$, respectively (see rules r_3 and r_5 in Fig. 2).

In order to compare the results of this subsection with the results of Ref. [9] let us write Eq. (3) as

$$h_i(t+\tau_{i,n})=h_i(t)+a(\delta_{i,j}w_i+\delta_{i+1,j}w_{i+1}+\delta_{i-1},w_{i-1}), \tag{13}$$

where

$$w_i=\theta(H_i^{i-1})\theta(H_i^{i+1}), \tag{14}$$

$$w_{i+1}=\theta^*(H_i^{i+1})\theta(H_{i+1}^{i+2})+\tfrac{1}{2}\theta^*(H_i^{i+1})\theta^*(H_{i+2}^{i+1}), \tag{15}$$

and

$$w_{i-1}=\theta^*(H_i^{i-1})\theta(H_{i-1}^{i-2})+\tfrac{1}{2}\theta^*(H_i^{i-1})\theta^*(H_{i-2}^{i-1}). \tag{16}$$

Developing $h_i(t+\tau_{i,n})$ in a Taylor series and retaining the first two terms, Eq. (8) can be written as

$$\frac{dh_i(t)}{dt}\,\tau_{i,n}=a(\delta_{i,j}w_i+\delta_{i+1,j}w_{i+1}+\delta_{i-1,j}w_{i-1}). \tag{17}$$

Using the results of Appendix A for the development in a Taylor series of the Kronecker symbols,

$$\delta_{i,j}=1+\eta_i(x_j)+\eta_i(x_i), \tag{18}$$

$$\delta_{i+1,j}=1+\eta_{i+1}(x_j)+\eta_{i+1}(x_{i+1}), \tag{19}$$

$$\delta_{i-1,j}=1+\eta_{i-1}(x_j)+\eta_{i-1}(x_{i-1}) \tag{20}$$

and the expression of $\tau_{i,n}$ given in Eq. (2), we obtain

$$\frac{dh_i(t)}{dt}=\frac{a}{\tau}(w_i+w_{i+1}+w_{i-1})+\eta. \tag{21}$$

The sum of the remainder terms η can be written as

$$\eta=\eta_0+\eta_s+\eta_d, \tag{22}$$

where

$$\eta_0=-\frac{dh_i(t)}{dt}\frac{\delta\tau_{i,n}}{\tau}, \tag{23}$$

$$\eta_s=\frac{\eta_i(x_j)w_i+\eta_{i+1}(x_j)w_{i+1}+\eta_{i-1}(x_j)w_{i-1}}{\tau}. \tag{24}$$

and

$$\eta_d=\frac{\eta_i(x_i)w_i+\eta_{i+1}(x_{i+1})w_{i+1}+\eta_{i-1}(x_{i-1})w_{i-1}}{\tau}. \tag{25}$$

The first term on the right-hand side of Eq. (21) is equal to the first transition moment given in Eq. (25) in Ref. [9].

Finally, by using of the results given in Appendix B and retaining the first few terms, we can write

$$\frac{\partial h(x_i,t)}{\partial t}=F+\nu\frac{\partial^2 h(x_i,t)}{\partial x_i^2}+\eta, \tag{26}$$

where

$$F=\frac{a}{\tau} \tag{27}$$

and

$$\nu=\frac{2a^3A_1}{\tau} \tag{28}$$

This is the well-known Edwards-Wilkinson equation [13].

B. Ballistic deposition

This model is defined by the following rules, shown graphically in Fig. 3:

$$h_i(t+\tau_{i,n})=h_i(t)+\frac{r_1+r_2+r_3+r_4+r_5}{W}, \tag{29}$$

where

$$r_1=\begin{cases} a & \text{if} \quad j=i \quad \text{and} \quad H_{i-1}^{i}\geqslant 0 \quad \text{and} \quad H_{i+1}^{i}\geqslant 0 \\ 0 & \text{if} \quad j\neq i \quad \text{or} \quad H_{i-1}^{i}<0 \quad \text{or} \quad H_{i+1}^{i}<0, \end{cases} \tag{30}$$

$$r_2=\begin{cases} H_i^{i+1} & \text{if} \quad j=i \quad \text{and} \quad H_i^{i-1}\geqslant 0 \quad \text{and} \quad H_{i-1}^{i+1}\geqslant 0 \\ 0 & \text{if} \quad j\neq i \quad \text{or} \quad H_i^{i-1}<0 \quad \text{or} \quad H_{i-1}^{i+1}<0, \end{cases} \tag{31}$$

$$r_3=\begin{cases} H_i^{i+1} & \text{if} \quad j=i \quad \text{and} \quad H_i^{i+1}>0 \quad \text{and} \quad H_{i-1}^{i}>0 \\ 0 & \text{if} \quad j\neq i \quad \text{or} \quad H_i^{i+1}\leqslant 0 \quad \text{or} \quad H_{i=1}^{i}<0, \end{cases} \tag{32}$$

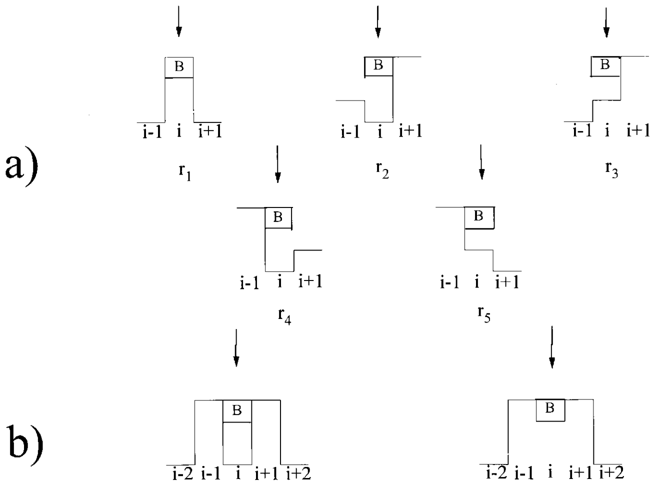

FIG. 3. (a) Growth rules corresponding to Eqs. (30), (31), (32), (33), and (34). (b) Growth rules corresponding to situations of multiple neighbors (two and three).

$$r_4=\begin{cases} H_i^{i-1} & \text{if} \quad j=i \quad \text{and} \quad H_i^{i+1}\geqslant 0 \quad \text{and} \quad H_{i+1}^{i-1}\geqslant 0 \\ 0 & \text{if} \quad j\neq i \quad \text{or} \quad H_i^{i+1}<0 \quad \text{or} \quad H_{i+1}^{i-1}<0, \end{cases} \tag{33}$$

$$r_5=\begin{cases} H_i^{i-1} & \text{if} \quad j=i \quad \text{and} \quad H_i^{i-1}>0 \quad \text{and} \quad H_{i+1}^{i}\geqslant 0 \\ 0 & \text{if} \quad j\neq i \quad \text{or} \quad H_i^{i-1}\leqslant 0 \quad \text{or} \quad H_{i+1}^{i}<0, \end{cases} \tag{34}$$

and

$$\begin{aligned} W=&\,\theta(H_{i-1}^{i})\,\theta(H_{i+1}^{i})+\theta(H_{i-1}^{i})\,\theta(H_{i-1}^{i+1}) \\ &+\theta^*(H_i^{i+1})\,\theta(H_{i-1}^{i})+\theta(H_i^{i+1})\,\theta(H_{i+1}^{i-1}) \\ &+\theta^*(H_i^{i-1})\,\theta(H_{i+1}^{i}) \end{aligned} \tag{35}$$

is a normalization factor that takes into account the situation of multiple neighbors as shown in Fig. 3(b). Again in Eq. (29) $h_i(t+\tau_{i,n}-\tau_0)=h_i(t)$ was used. In order to separate the deterministic and the stochastic parts of Eq. (29) let us write

$$h_i(t+\tau_{i,n})=h_i(t)+\delta_{i,j}\,\frac{(w_1+w_2+w_3)}{W}, \tag{36}$$

where

$$w_1=a\,\theta(H_{i-1}^{i})\,\theta(H_{i+1}^{i}), \tag{37}$$

$$w_2=H_i^{i+1}(\theta(H_i^{i-1})\,\theta(H_{i-1}^{i+1})+\theta^*(H_i^{i+1})\,\theta(H_{i-1}^{i})), \tag{38}$$

and

$$w_3=H_i^{i-1}(\theta(H_i^{i+1})\,\theta(H_{i+1}^{i-1})+\theta^*(H_i^{i-1})\,\theta(H_{i+1}^{i})). \tag{39}$$

Developing $h_i(t+\tau_{i,n})$ in a Taylor series and retaining the first two terms, Eq. (36) can be written as

$$\frac{dh_i(t)}{dt}\,\tau_{i,m}=\delta_{i,j}\,\frac{(w_1+w_2+w_3)}{W}. \tag{40}$$

Expanding the Kronecker symbol as in Appendix A and $\tau_{i,m}$ as in Eq. (2), Eq. (18) can be written as

$$\frac{dh_i(t)}{dt}=\frac{1}{\tau}\,\frac{(w_1+w_2+w_3)}{W}+\eta \tag{41}$$

and η as in the previous subsection is the sum of the remainder terms:

$$\eta=\eta_0+\eta_s+\eta_d, \tag{42}$$

where

$$\eta_0=-\frac{dh_i(t)}{dt}\,\frac{\delta\tau_{i,n}}{\tau}, \tag{43}$$

$$\eta_s=\frac{\eta_i(x_j)(w_1+w_2+w_3)}{W\tau}, \tag{44}$$

and

$$\eta_d=\frac{\eta_i(x_i)(w_1+w_2+w_3)}{W\tau}. \tag{45}$$

Finally using the results of Appendix B the Kardar-Parisi-Zhang equation is

$$\frac{\partial h(x_i,t)}{\partial t}=F+\nu\,\frac{\partial^2 h(x_i,t)}{\partial x_i^2}+\lambda\left(\frac{\partial h(x_i,t)}{\partial x_i}\right)^2+\eta, \tag{46}$$

where

$$F=\frac{a}{3\tau}, \tag{47}$$

$$\nu=\frac{[1-(5/6)aA_1]a^2}{3\tau}, \tag{48}$$

and

$$\lambda=\frac{a^2A_1}{\tau}. \tag{49}$$

In Eq. (46) the nonlinear term appears as the characteristic of the ballistic deposition model as expected.

III. GENERALIZATIONS AND CONCLUSIONS

The possible generalizations of the above method are the inclusion of different rules for deposition, diffusion, etc., and the generalization to higher dimensions and to lattices with other symmetries. In conclusion, this method of construction of the Langevin equations shows that discrete models are completely equivalent to the master equation approach and give the same stochastic Langevin equations. It is not just a new ''representation'' of the surface growth phenomena but a more powerful tool for solving models with vacancies and overhangs in a relatively simple way avoiding the algebraic complications arising in the master equation approach.

ACKNOWLEDGMENTS

The author wishes to acknowledge the Departamento de Fisica, Universidad Nacional de San Luis and the members of the Grupo de Fisicoquimica de Superficies, especially Dr. Victor D. Pereyra and Professor T. P. Eggarter for valuable suggestions throughout this work.

APPENDIX A

In order to separate the deterministic and the stochastic parts let us write the Kronecker symbol $\delta_{i,j}$ analytically as

$$\delta_{i,j} = \theta(x_i - x_j)\,\theta(-x_i + x_j), \tag{A1}$$

where $x_i = ia$ and $x_j = ja$. This way of writing $\delta_{i,j}$ allows one to separate the deterministic and the stochastic parts by approximating the step functions by an analytic function as in Ref. [9]. Expanding both step functions in a Taylor series it is easy to find

$$\theta(x_i - x_j) = \theta(x_i) + D^+, \tag{A2}$$

$$\theta(-x_i + x_j) = \theta(-x_i) + D^-, \tag{A3}$$

where

$$D^{\pm} = \sum_{n=1}^{\infty} \frac{(\pm x_j)^n}{(n)!} \left.\frac{\partial^n \theta(\pm x)}{\partial x^n}\right|_{x=\pm x_i}. \tag{A4}$$

Then Eq. (A1) can be written as

$$\delta_{i,j} = \theta(x_i)\,\theta(-x_i) + \eta_i(x_j), \tag{A5}$$

where

$$\eta_i(x_j) = \theta(x_i)D^- + \theta(-x_i)D^+ + D^- D^+. \tag{A6}$$

Developing again both step functions in Eqs. (A2) and (A3) about zero we have

$$\theta(\pm x_i) = 1 + D_0^{\pm}, \tag{A7}$$

where

$$D_0^{\pm} = \sum_{n=1}^{\infty} \frac{(\pm x_i)^n}{(n)!} \left.\frac{\partial^n \theta(\pm x)}{\partial x^n}\right|_{x=0}. \tag{A8}$$

Finally, introducing Eq. (A7) in Eq. (A5)

$$\delta_{i,j} = 1 + \eta_i(x_i) + \eta_i(x_j), \tag{A9}$$

where

$$\eta_i(x_i) = D_0^- + D_0^+ + D_0^- D_0^+. \tag{A10}$$

The generalization of Eq. (A9) is straightforward

$$\delta_{i\pm b,j} = 1 + \eta_{i\pm b}(x_{i\pm b}) + \eta_{i\pm b}(x_j) \tag{A11}$$

for any b and the explicit expressions are obtained after replacing everywhere the index i by $(i \pm b)$.

APPENDIX B

In order to obtain analytic expressions of the height differences of two sites we define an interpolating function $h(x_i,t)$ such that

$$h(x_i,t) = h_i(t) \quad \text{for} \quad i = 1,2,\ldots,N. \tag{B1}$$

Now the height differences can be written as

$$H_i^{i\pm1} = h(x_{i\pm1},t) - h(x_i,t) \tag{B2}$$

and

$$H_{i\pm1}^{i\pm2} = h(x_{i\pm2},t) - h(x_{i\pm1},t). \tag{B3}$$

Finally, developing in a Taylor series all heights about x_i it is easy to write the height differences of the previous equation as

$$h(x_{i\pm1},t) - h(x_i,t) = \pm D_{1,1} + D_{2,1} \tag{B4}$$

and

$$h(x_{i\pm2},t) - h(x_{i\pm1},t) = \pm(D_{1,2} - D_{1,1}) + (D_{2,2} - D_{2,1}), \tag{B5}$$

where

$$D_{1,k} = \sum_{n=1}^{\infty} \frac{(ka)^{2n+1}}{(2n+1)!} \frac{\partial^{2n+1} h(x_i,t)}{\partial x_i^{2n+1}} \quad \text{for} \quad k = 1,2 \tag{B6}$$

and

$$D_{2,k} = \sum_{n=1}^{\infty} \frac{(ka)^{2n}}{(2n)!} \frac{\partial^{2n} h(x_i,t)}{\partial x_i^{2n}} \quad \text{for} \quad k = 1,2. \tag{B7}$$

Using the above developments we proceed to find the Langevin equations of the discrete models.

1. Das Sarma–Tamborenea model (without diffusion)

Let us approximate the step functions by analytic functions as in Ref. [9] and, expanding each one in a Taylor series, we easily find

$$w_i = 1 + 2A_1 D_{2,1} + O(A_1^2), \tag{B8}$$

$$w_{i+1} = A_1(D_{1,1} + D_{2,1}) + O(A_1^2), \tag{B9}$$

and

$$w_{i-1} = A_1(-D_{1,1} + D_{2,1}) + O(A_1^2), \tag{B10}$$

where A_1 is the second coefficient of the development and only the terms $O(A_1)$ are retained. The developments of the height differences given in Appendix A were also used. Finally, collecting together the above results one can be written

$$w_i + w_{i+1} + w_{i-1} = 1 + 4A_1 D_{2,1} + O(A_1^2), \tag{B11}$$

which is the deterministic part of the Langevin equation corresponding to the Das Sarma–Tamborenea model.

2. Ballistic deposition

In the same way that we proceeded in the previous subsection the three terms of the numerator of the deterministic part of the ballistic deposition model are

$$w_1 = a(1-2A_1D_{2,1}) + O(A_1^2), \tag{B12}$$

$$w_2 = D_{1,1} + D_{2,1} + 2A_1(D_{1,1}+D_{2,1})^2 + O(A_1^2), \tag{B13}$$

and

$$w_3 = -D_{1,1} + D_{2,1} + 2A_1(-D_{1,1}+D_{2,1})^2 + O(A_1^2), \tag{B14}$$

where again only the terms $O(A_1)$ are retained and the development of the height differences given in Appendix A was also used. Let

$$(-D_{1,1}+D_{2,1})^2 = D_{1,1}^2 + O(a^3) \tag{B15}$$

in the above results. Then the numerator of the deterministic part of the Langevin equation for the ballistic deposition model can be written as

$$w_1+w_2+w_3 = a + \left(1-\frac{aA_1}{2}\right)2D_{2,1} + 3A_1D_{1,1}^2 + O(A_1^2). \tag{B16}$$

In the same way the denominator is

$$W = 3 + 2A_1D_{2,1} + O(A_1^2). \tag{B16}$$

Expanding $1/W$ in a geometric series we finally obtain

$$\frac{1}{W} = \frac{1}{3}\left[1-\tfrac{2}{3}A_1D_{2,1} + O(A_1^2)\right] \tag{B17}$$

and

$$\frac{w_1+w_2+w_3}{W} = \frac{a}{3} + \left(1-\frac{5aA_1}{6}\right)2D_{2,1} + A_1D_{1,1}^2 + O(A_1^2), \tag{B18}$$

which is the deterministic part of the ballistic deposition model.

[1] A.-L. Barabasi and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, New York, 1995).
[2] D. N. Sutherland, J. Colloid. Interface Sci. **22**, 300 (1966).
[3] M. J. Vold, J. Colloid. Sci. **14**, 168 (1959).
[4] M. J. Vold, J. Phys. Chem. **63**, 1608 (1959).
[5] R. Baiod, D. Kessler, P. Ramanlal, L. Sander, and R. Savit, Phys. Rev. A **38**, 3672 (1988).
[6] F. Family and T. Vicsek, J. Phys. A **18**, L75 (1985).
[7] P. Meakin and R. Jullien, SPIE, Proc. Int. Soc. Opt. Eng. **821**, 45 (1987).
[8] M. A. Herman and H. Sitter, *Molecular Beam Epitaxy: Fundamentals and Currents Status* (Springer-Verlag, Berlin, 1989).
[9] D. D. Vvdensky, A. Zangwill, C. N. Luse, and M. R. Wilby, Phys. Rev. E **48**, 852 (1993).
[10] F. Family, J. Phys. A **19**, 441 (1986).
[11] D. E. Wolf and J. Villain, Europhys. Lett. **13**, 389 (1990).
[12] S. Das Sarma and P. Tamborenea, Phys. Rev. Lett. **66**, 325 (1991).
[13] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London A **381**, 17 (1982).
[14] M. Kardar, G. Parisi, and I. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).